Journal of Approximation Theory **100**, 284–303 (1999)
Article ID jath.1999.3347, available online at http://www.idealibrary.com on IDEAL®

Best Uniform Approximation of Complex-Valued Functions by Generalized Polynomials Having Restricted Ranges

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Communicated by Manfred v. Golitschek

Received April 15, 1997; accepted in revised form January 15, 1999

We study the uniform best restricted ranges approximations of complex-valued functions by generalized polynomials. The theory, generalizing the real-valued case, embraces the theorems of existence, characterization, uniqueness, and strong uniqueness.

1. INTRODUCTION

The problems of best uniform restricted ranges approximation have been thoroughly studied in the framework of the well-established theory of best constrained approximation of real-valued functions (see the corresponding review in [1] and the relevant references therein; a modern approach to the problem is presented in [2]).

In this article we consider the problem of best uniform restricted ranges approximation of *complex-valued* continuous functions, which in analogy with the real-valued case [3, 4] can be formulated as follows. *Let $C(Q)$ be the space of continuous complex-valued functions defined on a compact set Q, let $P \subset C(Q)$ be a finite-dimensional subspace in it, and let $\Omega = \{\Omega_t \mid t \in Q\}$*

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0021-9045/99 $30.00

be a system of non-empty convex and closed sets in $\mathbb{C}$. *For a given function* $f \in C(Q)$ *set*

$$E(f) := \inf_{p \in P_\Omega} \|f - p\|, \tag{1.1}$$

where

$$P_\Omega := \{p \in P \mid p(t) \in \Omega_t \textit{ for all } t \in Q\}.$$

Here $\| \ \|$ *stands for the uniform norm.*

The problem is to investigate the properties of the elements $p^* \in P_\Omega$ *providing the* infimum *in* (1.1). Admittedly, this problem for a general class of restriction is quite difficult.

In this work the problems of existence, characterization, uniqueness and strong uniqueness of such an element p^* are studied for some special system of restrictions Ω, using the notion of a *minimal admissible pair of sets* corresponding to the notion of a characterization set of best approximation (see, for instance, [5]) in the classical theory of uniform approximation.

The organization of this paper is as follows. In Section 2 we introduce the basic definitions, notations, and facts to be employed throughout the article. We also present the theorem on existence of best restricted ranges approximation. The definition and properties of a minimal admissible pair of sets constitute the subject of Section 3. We present the three criteria of best approximation (including the *Kolmogorov-type characterization* and *zero in the convex hull characterization*) in Section 4. In Section 5 the theorems of uniqueness and strong uniqueness of best approximation and the theorem on continuity of the operator of best approximation are proved. In Section 6 we make concluding remarks.

2. BASIC DEFINITIONS, NOTATIONS AND FACTS

Let Q be a compact set in the complex plane $\mathbb{C}$ containing at least $n+1$ points. Denote by $C(Q)$ the Banach algebra of all complex-valued continuous functions defined on Q with the norm

$$\|f\| = \max_{t \in Q} |f(t)|.$$

For every function $f \in C(Q)$ introduce the set $M(f)$

$$M(f) := \{t \in Q \mid |f(t)| = \|f\|\}.$$

Clearly, $M(f)$ is compact. Consider an n-dimensional subspace $P \subset C(Q)$ with a basis $\{\varphi_1, \varphi_2, ..., \varphi_n\}$. The elements $p \in P$ have the form

$$p = \sum_{v=1}^{n} c_v \varphi_v,$$

where $c_v \in \mathbb{C}$, $v = 1, ..., n$. We call them *generalized polynomials with respect to the system* $\{\varphi_1, \varphi_2, ..., \varphi_n\}$, or just *polynomials*, for short. For $p \in P$ set

$$Z(p) := \{t \in Q \mid p(t) = 0\}.$$

DEFINITION 2.1 [6]. An n-dimensional subspace $P \subset C(Q)$ is called a *Haar space* if every polynomial $p \in P \setminus \{0\}$ has no more than $n-1$ zeros in Q.

Let $u \in C(Q)$ and $r \in C(Q)$ be fixed functions, in addition assume that $r(t) > 0$ for all $t \in Q$. For every point $t \in Q$ denote

$$\Omega_t := \{z \in \mathbb{C} \mid |z - u(t)| \leqslant r(t)\},$$

$$\operatorname{int} \Omega_t := \{z \in \mathbb{C} \mid |z - u(t)| < r(t)\},$$

$$\partial\Omega_t := \{z \in \mathbb{C} \mid |z - u(t)| = r(t)\}.$$

HYPOTHESIS 2.1. *Throughout this paper we assume that always for some* $p_0 \in P$ *the condition*

$$p_0(t) \in \operatorname{int} \Omega_t$$

holds for all $t \in Q$.

For all $p \in P$ set

$$B(p) := \{t \in Q \mid p(t) \in \partial\Omega_t\}.$$

In view of continuity of the functions u, r and p the set $B(p)$ is compact. Introduce the notation

$$P_{B,\Omega} := \{p \in P \mid p(t) \in \Omega_t \text{ for all } t \in B\},$$

where $B \subset Q$, $P_{\varnothing,\Omega} := P$, $P_{Q,\Omega} = P_\Omega$. Note that for every set $B \subset Q$ the set $P_{B,\Omega}$ is convex, while for a closed set B the set $P_{B,\Omega}$ is closed in P. The inclusion $B' \subset B$ obviously implies $P_{B,\Omega} \subset P_{B',\Omega}$.

Let $\mathfrak{M}$ be the set of ordered pairs $(A; B)$, where $A \subset Q$, $B \subset Q$ and $A \neq \emptyset$. We write $(A'; B') \subset (A; B)$ iff $A' \subset A$ and $B' \subset B$. Then the inclusion $(A'; B') \subset (A; B)$ is called *strict*, if at least one of the inclusions $A' \subset A$ and $B' \subset B$ is strict.

For a function $f \in C(Q)$ and a pair $(A; B) \in \mathfrak{M}$ set

$$E_A(f; P_{B,\Omega}) := \inf_{p \in P_{B,\Omega}} \sup_{t \in A} |f(t) - p(t)|.$$

Clearly, for $A = B = Q$,

$$E_Q(f; P_{Q,\Omega}) = E_Q(f; P_\Omega) = E(f).$$

It is easily seen that the inclusion $(A'; B') \subset (A; B)$ implies the inequality

$$E_{A'}(f; P_{B',\Omega}) \leqslant E_A(f; P_{B,\Omega}),$$

which leads, in particular, to

$$E_A(f; P_{B,\Omega}) \leqslant E(f)$$

for any pair $(A; B) \in \mathfrak{M}$.

Definition 2.2. A polynomial $q \in P_{B,\Omega}$, satisfying the equality

$$\sup_{t \in A} |f(t) - q(t)| = E_A(f; P_{B,\Omega}),$$

is called a *best restricted ranges approximation to f on A from $P_{B,\Omega}$*.

A best restricted ranges approximation to f on Q from P_Ω, or the polynomial $p^* \in P_\Omega$ satisfying

$$\|f - p^*\| = E(f)$$

is called for short a *best approximation to f from P_Ω*.

The compactness argument justifies the validity of the following

Theorem 2.1. *If A and B are compact subsets of Q $(A \neq \emptyset)$, then for every function $f \in C(Q)$ there exists a best restricted ranges approximation to f on A from $P_{B,\Omega}$.*

Corollary 2.1. *For every function $f \in C(Q)$ there exists a best approximation to f from P_Ω.*

Next, let us formulate in the complex form the following three classical results.

THEOREM 2.2 (On Linear Inequalities [7]). *Let U be a compact subset of $\mathbb{C}^n$. Then there exists a point $\mathbf{z} \in \mathbb{C}^n$ such that* $\operatorname{Re}(\mathbf{z}, \mathbf{u}) > 0$ *for all $\mathbf{u} \in U$ iff the origin of $\mathbb{C}^n$ does not belong to the convex hull of U.*

Here (,) means the scalar product in $\mathbb{C}^n$.

THEOREM 2.3 (Carathéodory [7]). *Let A be a subset of an n-dimensional complex space. Every point of the convex hull of A is expressible in the form of a convex linear combination of $2n+1$ (or fewer) elements of A.*

THEOREM 2.4 (Helly [12]). *Let $\{V\}$ be a collection of closed and convex sets V in $\mathbb{C}^n$ such that every $2n+1$ among them have a common point. Then all the sets V have a common point, provided that there exists a finite subcollection $V_1, V_2, ..., V_s (s \geqslant 1)$ of elements of $\{V\}$, such that their intersection $V_1 \cap V_2 \cap \cdots \cap V_s$ is non-void and bounded.*

Throughout this article $|A|$ denotes the cardinality of a set A.

3. MINIMAL ADMISSIBLE PAIRS OF SETS AND THEIR PROPERTIES

Let $f \in C(Q)$.

DEFINITION 3.1. An ordered pair $(A; B) \in \mathfrak{M}$ is called an *admissible pair* (*a.p.*) *for a function f with respect to P_Ω*, if

$$E_A(f; P_{B, \Omega}) = E(f).$$

DEFINITION 3.2. An admissible pair $(A_0; B_0)$ for f with respect to P_Ω is called a *minimal admissible pair* (*m.a.p.*) *for a function f* with respect to P_Ω, if the *strict* inclusion $(A; B) \subset (A_0; B_0)$ implies the strict inequality

$$E_A(f; P_{B, \Omega}) < E_{A_0}(f; P_{B_0, \Omega}). \tag{3.1}$$

Remark 3.1. Each a.p. $(A; B)$ for a function f, where A and B are finite subsets of Q, admits at least one m.a.p. for f.

THEOREM 3.1. *Let $(A_0; B_0) \in \mathfrak{M}$ be a m.a.p. for $f \in C(Q)$ with respect to P_Ω, and $p^* \in P_\Omega$ be a best approximation to f from P_Ω. Then simultaneously the following inclusions hold*:

$$A_0 \subset M(f - p^*), \qquad B_0 \subset B(p^*). \tag{3.2}$$

Proof. By contradiction:

(a) Assume that the first inclusion of (3.2) does not hold. Then, there exists a point $t_0 \in A_0$, a polynomial $\tilde{p} \in P_{B_0, \Omega}$, a positive constants δ_1, δ_2 (see Definition 3.2) such that

$$|f(t_0) - p^*(t_0)| = E(f) - \delta_1, \tag{3.3}$$

$$\sup_{t \in A_0 \setminus \{t_0\}} |f(t) - \tilde{p}(t)| = E(f) - \delta_2. \tag{3.4}$$

For an arbitrary $\lambda \in (0, 1)$ consider a polynomial p_λ of the form

$$p_\lambda := (1 - \lambda)\, p^* + \lambda \tilde{p}.$$

Taking into account convexity of the set $P_{B_0, \Omega}$ and the inclusions $\tilde{p} \in P_{B_0}$, Ω, $p^* \in P_\Omega \subset P_{B_0, \Omega}$ we get

$$p_\lambda \in P_{B_0, \Omega} \qquad \text{for any} \quad \lambda \in (0, 1).$$

Using (3.3), we get

$$|f(t_0) - p_\lambda| < E(f) - \tfrac{1}{2}\delta_1 \tag{3.5}$$

for small enough parameters $\lambda \in (0, 1)$. For each point $t \in A_0 \setminus \{t_0\}$ and an arbitrary $\lambda \in (0, 1)$, using (3.4), we have

$$f(t) - p_\lambda(t_0)| < E(f) - \tfrac{1}{2}\delta_1 \tag{3.6}$$

(note that in (3.6) we write t, but not t_0). Employing the inequalities (3.5) and (3.6), we derive for small enough $\lambda \in (0, 1)$ the estimation

$$E_{A_0}(f; P_{B_0, \Omega}) \leqslant \sup_{t \in A_0} |f(t) - p_\lambda(t)| < E(f),$$

which is impossible, since $(A_0; B_0)$ is a m.a.p. for f. Hence $A_0 \subset M(f - p^*)$.

(b) Assume now that the inclusion $B_0 \subset B(p^*)$ does not hold true. Then, there exists a point $t_0 \in B(p^*)$, for which

$$p^*(t_0) \in \operatorname{int} \Omega_{t_0},$$

that is,

$$|p^*(t_0) - u(t)| < r(t_0).$$

In view of Definition 3.2 one can find a polynomial $\tilde{q} \in P_{B_0 \setminus \{t_0\}, \Omega}$, such that

$$\sup_{t \in A_0} |f(t) - \tilde{q}(t)| < E(f).$$

Repeating the technique of the part (a), we can show that for a small enough parameter $\lambda \in (0, 1)$ the polynomial

$$q_\lambda := (1-\lambda)\, p^* + \lambda \tilde{q} \in P_{B_0, \Omega};$$

in addition

$$\sup_{t \in A_0} |f(t) - q_\lambda(t)| < E(f),$$

which is impossible for the m.a.p. (A_0, B_0). Hence, $B_0 \subset B(p^*)$, as was to be proved. ▮

THEOREM 3.2. *For each function $f \in C(Q)$ there exists at least one m.a.p. $(A_0; B_0)$ for f with respect to P_Ω, such that*

$$|A_0 \cup B_0| \leqslant 2n+1.$$

Proof. Taking into account Remark 3.1, it is enough to show that for some set $D_0 \subset Q$ with $|D_0| \leqslant 2n+1$ the pair (D_0, D_0) is an a.p. for f. Carry out the proof in a few steps.

(a) The subsets $D = \{t_1, t_2, ..., t_{2n+1}\}$ of Q with $2n+1$ points (with possible repetitions) can be interpreted as points $\mathbf{t} = (t_1, t_2, ..., t_{2n+1})$ in the product space Q^{2n+1}. Introduce an auxiliary function $\Phi : Q^{2n+1} \to \mathbb{R}$ by setting for each point $\mathbf{t} = (t_1, t_2, ..., t_{2n+1}) \in Q^{2n+1}$,

$$\Phi(\mathbf{t}) = E_D(f; P_{D, \Omega}),$$

where $D = \{t_1, t_2, ..., t_{2n+1}\} \subset Q$. It is easily seen that for each point $\mathbf{t} \in Q^{2n+1}$ the conditional inequality holds true:

$$\Phi(\mathbf{t}) \leqslant E(f). \tag{3.7}$$

(b) Let us prove that the function Φ is continuous from above on Q^{2n+1}. Fix an arbitrary point $\mathbf{t}' = (t'_1, t'_2, ..., t'_{2n+1}) \in Q^{2n+1}$ (simultaneously setting $D' = \{t'_1, t'_2, ..., t'_{2n+1}\}$) and an arbitrary number $\varepsilon > 0$. According to Theorem 2.1 there exists a polynomial $p' \in P_{D', \Omega}$ such that

$$|f(t'_k) - p'(t'_k)| \leqslant \Phi(\mathbf{t}'), \qquad k = 1, ..., 2n+1.$$

Define $p'' := (1-\lambda)\, p' + \lambda p_0$, where p_0 is a polynomial from Hypothesis 2.1. It is understood that for some small enough $\lambda \in (0, 1)$ we have

$$|f(t'_k) - p''(t'_k)| < \Phi(\mathbf{t}') + \varepsilon \tag{3.8}$$

$$|u(t'_k) - p''(t'_k)| < r(t'_k) \tag{3.9}$$

$$k = 1, ..., 2n+1.$$

In view of the inequalities (3.8) and (3.9) and continuity on Q of the functions f, p'', u, r, for each $k = 1, ..., 2n+1$ there exists a neighborhood O_k of the point t'_k such that for all $t_k \in O_k$ the following inequalities hold:

$$|f(t_k) - p''(t_k)| < \Phi(\mathbf{t}') + \varepsilon, \qquad |u(t_k) - p''(t_k)| < r(t_k). \tag{3.10}$$

Taking an arbitrary point $\mathbf{t} = (t_1, t_2, ..., t_{2n+1}) \in O_1 \times O_2 \times \cdots \times O_{2n+1}$, the corresponding set $D = \{t_1, t_2, ..., t_{2n+1}\}$, we come to the conclusion that the polynomial p'' belongs to the set $P_{D,\Omega}$ and the following inequality holds:

$$\Phi(\mathbf{t}) = E_D(f; P_{D,\Omega}) \leqslant \max_{1 \leqslant k \leqslant 2n+1} |f(t_k) - p''(t_k)| < \Phi(\mathbf{t}') + \varepsilon.$$

Therefore the function Φ is continuous from above at an arbitrary point $\mathbf{t}' \in Q^{2n+1}$, or everywhere on Q^{2n+1}.

(c) By Weierstrass' theorem there always exists such a point $\mathbf{t}^0 \in Q^{2n+1}$ with the corresponding set $D_0 \subset Q$ that

$$E_{D_0}(f; P_{D_0,\Omega}) = \Phi(\mathbf{t}^0) = \max_{\mathbf{t} \in Q^{2n+1}} \Phi(\mathbf{t}) =: E_0 \tag{3.11}$$

Note that $|D_0| \leqslant 2n+1$. Moreover, it follows from Theorem 2.1 that for each set $D = \{t_1, t_2, ..., t_{2n+1}\} \subset Q$ and the corresponding point $\mathbf{t} = (t_1, t_2, ..., t_{2n+1}) \in Q^{2n+1}$ there exists a polynomial $p \in P_{D,\Omega}$ such that the following inequalities hold:

$$|f(t_k) - p(t_k)| \leqslant \Phi(\mathbf{t}) \leqslant \Phi(\mathbf{t}^0) = E_0. \tag{3.12}$$

(d) We prove, using Helly's theorem, that the pair (D_0, D_0) is an a.p. for f. Indeed, introduce for each point $t \in Q$ the set

$$V_t := \{p \in P \,|\, |f(t) - p(t)| \leqslant E_0 \text{ and } p(t) \in \Omega_t\}.$$

Notice that each set V_t is convex and closed. In addition, by virtue of (3.12), arbitrary $2n+1$ sets V_t have a common point. Next, linear independence of the system $\{\phi_1, \phi_2, ..., \phi_n\}$ entails that there is a set of points $\{\hat{t}_1, \hat{t}_2, ..., \hat{t}_n\} \subset Q$ such that $\det[\phi_l(\hat{t}_j)]_{i,j=1}^n \neq 0$. For each $p \in P$ define

$$\Lambda_1(p) := \max_{1 \leqslant l \leqslant n} |p(\hat{t}_l)|.$$

It is easy to show that $\Lambda_1(\cdot)$ is a norm on P. Since all norms on P are equivalent, for some $\mu > 0$ and for each $p \in P$ we have

$$\|p\| \leqslant \mu \Lambda_1(p). \tag{3.13}$$

Now for each polynomial $p \in \bigcap_{l=1}^n V_{\hat{t}_l}$ in view of (3.13) we have the following estimation

$$\|p\| \leqslant \mu \Lambda_1(p) = \mu \max_{1 \leqslant l \leqslant n} |p(\hat{t}_l)| \leqslant \mu\Big(\max_{1 \leqslant l \leqslant n} |p(\hat{t}_l) - f(\hat{t}_l)| + \max_{1 \leqslant l \leqslant n} |f(\hat{t}_l)|$$

$$\leqslant \mu(E_0 + \|f\|).$$

Hence, the set $\bigcap_{l=1}^n V_{\hat{t}_l}$ is bounded. The isomorphism between P and $\mathbb{C}^n$, by Helly's theorem, entails that all the sets V_t have a common point.

Let $\tilde{p}_0 \in \bigcap_{t \in Q} V_t$. Then for all $t \in Q$ the following inclusion holds: $\tilde{p}_0(t) \in \Omega_t$; in addition

$$|f(t) - \tilde{p}_0(t)| \leqslant E_0.$$

Thus, $\tilde{p}_0 \in P_\Omega$, which leads (taking into account (3.7)) to

$$E(f) \leqslant \|f - \tilde{p}_0\| \leqslant E_0 = \Phi(\mathbf{t}^0) \leqslant E(f).$$

Finally,

$$E(f) = E_0 = E_{D_0}(f; P_{D_0, \Omega}).$$

This completes the proof. ▮

Definition 3.3. We call a function $f \in C(Q)$ *admissible*, if it satisfies at least either of the two conditions

$$(1) \quad f(t) \in \Omega_t \qquad \text{for all} \quad t \in Q;$$

$$(2) \quad M(f - p^*) \cap B(p^*) = \varnothing,$$

where $p^* \in P_\Omega$ is some best approximation to f from P_Ω.

We denote the set of all admissible functions by $C_a(Q)$.

THEOREM 3.3. *Let P be a Haar space and $f \in C_a(Q) \backslash P_\Omega$. Then each m.a.p. $(A_0; B_0)$ for the function f with respect to P_Ω satisfies the condition*

$$|A_0 \cup B_0| \geqslant n+1.$$

Proof. First of all notice that for every set consisting of n distinct points $\{t_1, t_2, ..., t_n\} \subset Q$ and an arbitrary set of numbers $\{c_1, c_2, ..., c_n\} \subset \mathbb{C}$ there exists in the Haar space P a polynomial p satisfying (see [6], p. 68)

$$p(t_k) = c_k, \qquad k = 1, ..., n.$$

We continue by contradiction. Assume that for some m.a.p. (A_0, B_0) for f the conditional inequality holds $|A_0 \cup B_0| \leqslant n$. Now consider in accordance with Definition 3.3 two cases:

(a) Let $f(t) \in \Omega_t$ for each $t \in Q$. Set

$$C_0 := A_0 \cup B_0 = \{t_1, ..., t_k\}, \qquad k \leqslant n.$$

We complete if needed the set C_0 up to a set of n points and consider a polynomial $\tilde{p} \in P$ satisfying

$$\tilde{p}(t_\ell) = f(t_\ell), \qquad \ell = 1, ..., k.$$

Then, obviously $\tilde{p} \in P_{C_0, \Omega} \subset P_{B_0, \Omega}$ and so

$$E_{A_0}(f; P_{B_0, \Omega}) \leqslant E_{C_0}(f; P_{C_0, \Omega}) \leqslant \max_{1 \leqslant l \leqslant k} |f(t_\ell) - \tilde{p}(t_\ell)| = 0 < E(f),$$

since $f \notin P_\Omega$, which contradicts the definition of a m.a.p.

(b) Assume that for some best approximation $p^* \in P_\Omega$ to the function f we have the condition $M(f - p^*) \cap B(p^*) = \varnothing$. Due to Theorem 3.1 the following inclusions hold: $A_0 \subset M(f - p^*)$, $B_0 \subset B(p^*)$. Therefore $A_0 \cap B_0 = \varnothing$. Let $A_0 = \{t_1, ..., t_s\}$, $B_0 = \{t_{s+1}, ..., t_k\}$, $k \leqslant n$. Choose such a polynomial $\tilde{p}$ in P that

$$\tilde{p}(t_\ell) = f(t_\ell), \qquad \ell = 1, ..., s,$$

$$\tilde{p}(t_\ell) = p^*(t_\ell), \qquad \ell = s+1, ..., k.$$

Due to the obvious inclusion $\tilde{p} \in P_{B_0, \Omega}$ we have the estimation

$$E_{A_0}(f; P_{B_0, \Omega}) \leqslant \max_{t \in A_0} |f(t) - \tilde{p}(t)| = 0 < E(f)$$

since $f \notin P_\Omega$, which is impossible for a m.a.p. This completes the proof. ∎

4. CHARACTERIZATION OF BEST APPROXIMATION

Let $f \in C(Q)$, $p^* \in P_\Omega$. Set

$$\sigma_1(t) := f(t) - p^*(t), \qquad t \in M(f - p^*),$$
$$\sigma_2(t) := u(t) - p^*(t), \qquad t \in B(p^*).$$

THEOREM 4.1 (Kolmogorov-Type Characterization). *A polynomial $p^* \in P_\Omega$ is a best approximation to a function $f \in C(Q)$ from P_Ω, if and only if for each $p \in P$ the following conditional inequality holds true*:

$$\min\{\min_{t \in M(f-p^*)} \operatorname{Re}(p(t)\,\overline{\sigma_1(t)}),\ \min_{t \in B(p^*)} \operatorname{Re}(p(t)\,\overline{\sigma_2(t)})\} \leqslant 0. \qquad (4.14)$$

Proof. $\Rightarrow$ In the case of f belonging to P_Ω we have $\sigma_1(t) = f(t) - p^*(t) = 0$ for all $t \in Q$, and so (4.14) is true. Let $f \in C(Q) \backslash P_\Omega$. We proceed by contradiction. Assume that for some polynomial $q \in P_\Omega$ the condition (4.14) does not hold, that is, we have the inequalities

$$\begin{aligned} &\operatorname{Re}(q(t)\,\overline{\sigma_1(t)}) > 0, \qquad t \in M(f - p^*), \\ &\operatorname{Re}(q(t)\,\overline{\sigma_2(t)}) > 0, \qquad t \in B(p^*) \end{aligned} \qquad (4.15)$$

By virtue of Theorem 3.2 there exists such a m.a.p. $(A_0; B_0)$ for f that $|A_0 \cup B_0| \leqslant 2n+1$. Moreover, in view of Theorem 3.1 we have the inclusions

$$A_0 \subset M(f - p^*), \qquad B_0 \subset B(p^*),$$

leading along with the inequalities (4.15) to

$$\begin{aligned} &\operatorname{Re}(q(t)\,\overline{\sigma_1(t)}) > 0, \qquad t \in A_0, \\ &\operatorname{Re}(q(t)\,\overline{\sigma_2(t)}) > 0, \qquad t \in B_0. \end{aligned} \qquad (4.16)$$

Taking into account that both A_0 and B_0 are finite sets, we introduce the constant λ_0,

$$\lambda_0 := \min\left\{\min_{t \in A_0} \frac{2\operatorname{Re}(q(t)\,\overline{\sigma_1(t)})}{|q(t)|^2},\ \min_{t \in B_0} \frac{2\operatorname{Re}(q(t)\,\overline{\sigma_2(t)})}{|q(t)|^2}\right\}.$$

Notice that in view of (4.16), $\lambda_0 > 0$. Now for a fixed $\lambda \in (0, \lambda_0)$ and an arbitrary point $t \in B_0$ we have

$$\begin{aligned} |u(t) - p^*(t) - \lambda q(t)|^2 &= |u(t) - p^*(t)|^2 - 2\lambda \operatorname{Re}(q(t)\,\overline{\sigma_2(t)}) + \lambda^2\,|q(t)|^2 \\ &= r^2(t) + \lambda\,|q(t)|^2 \left(\lambda - \frac{2\operatorname{Re}(q(t)\,\overline{\sigma_2(t)})}{|q(t)|^2}\right) < r^2(t). \end{aligned}$$

Therefore $p^* + \lambda q \in P_{B_0, \Omega}$. We can show in an analogous way that for each point $t \in A_0$ and the same $\lambda \in (0, \lambda_0)$ the following inequalities hold:

$$|f(t) - p^*(t) - \lambda q(t)|^2 < |f(t) - p^*(t)|^2 = \|f - p^*\|^2 = E^2(f).$$

Finally, we get

$$E_{A_0}(f; P_{B_0, \Omega}) \leqslant \max_{t \in A_0} |f(t) - p^*(t) - \lambda q(t)| < E(f),$$

which is impossible for the m.a.p. (A_0, B_0). The obtained contradiction proves the 'if' part of the theorem.

$\Leftarrow$ Suppose for every polynomial $p \in P$ the condition (4.14) holds. Fix an arbitrary polynomial $q \in P_\Omega$ and for an arbitrary $\lambda \in (0, 1)$ set $q_\lambda := (1 - \lambda) q + \lambda p_0$, where p_0 is the polynomial of Hypothesis 2.1. Then, clearly, for all points $t \in Q$ (in particular, for $t \in B(p^*)$) we have the inclusion $q_\lambda \in \operatorname{int} \Omega_t$, hence the absolute inequalities

$$|u(t) - q_\lambda(t)| < r(t) = |u(t) - p^*(t)|, \qquad t \in B(p^*), \quad \lambda \in (0, 1),$$

hold, leading, after simple transformations, to

$$\operatorname{Re}((q_\lambda(t) - p^*(t)) \overline{\sigma_2(t)}) > 0$$

for all $t \in B(p^*)$ and $\lambda \in (0, 1)$. But then due to (4.14) for the polynomial $q_\lambda - p^*$ there exists such a point $t_\lambda \in M(f - p^*)$ that

$$\operatorname{Re}((q_\lambda(t_\lambda) - p^*(t_\lambda)) \overline{\sigma_1(t_\lambda)}) \leqslant 0.$$

Hence continuing, we derive the following chain of inequalities

$$\begin{aligned} \|f - p^*\|^2 = |f(t_\lambda) - p^*(t_\lambda)|^2 &= \operatorname{Re}(f(t_\lambda) - p^*(t_\lambda)) \overline{\sigma_1(t_\lambda)}) \\ &\leqslant \operatorname{Re}(f(t_\lambda) - q_\lambda(t_\lambda)) \overline{\sigma_1(t_\lambda)}) \\ &\leqslant |f(t_\lambda) - q_\lambda(t_\lambda)| \cdot |f(t_\lambda) - p^*(t_\lambda)| \leqslant \|f - q_\lambda\| \cdot \|f - p^*\|. \end{aligned}$$

Thus, for each $\lambda \in (0, 1)$ we have

$$\|f - p^*\| \leqslant \|f - q_\lambda\|.$$

By passing to the limit in the last inequality as $\lambda \to +0$, we obtain the inequality

$$\|f - p^*\| \leqslant \|f - q\| \qquad \text{for all} \quad q \in P_\Omega.$$

Therefore p^* is a best approximation to f from P_Ω, which was to be proved. ▌

For each function $f \in C(Q)$ and $p^* \in P_\Omega$ consider the set

$$\mathscr{B} = \{\mathbf{b}(t) = (\overline{\varphi_1(t)}, \overline{\varphi_2(t)}, ..., \overline{\varphi_n(t)})\, \sigma_1(t) \mid t \in M(f - p^*)\}$$
$$\cup \{\mathbf{c}(t) = (\overline{\varphi_1(t)}, \overline{\varphi_2(t)}, ..., \overline{\varphi_n(t)})\, \sigma_2(t) \mid t \in B(p^*)\},$$

noticing that due to compactness of the sets $M(f - p^*)$ and $B(p^*)$ in Q the set $\mathscr{B}$ is compact in $\mathbb{C}^n$.

THEOREM 4.2 ("Zero in the Convex Hull" Characterization). *A polynomial $p^* \in P_\Omega$ is a best approximation to a function $f \in C(Q) \backslash P_\Omega$ if and only if the origin of the space $\mathbb{C}^n$ belongs to the convex hull of $\mathscr{B}$.*

Proof. Consider an arbitrary polynomial $p \in P$ in the form $p = \sum_{\nu=1}^{n} c_\nu \phi_\nu$ and the corresponding vector $\mathbf{z} = (c_1, c_2, ..., c_n) \in \mathbb{C}^n$. Let $p^* \in P_\Omega$ is a best approximation to $f \in C(Q) \backslash P_\Omega$. In view of Theorem 4.1 it is equivalent to the fact that for each polynomial $p \in P$ at least either of the inequalities

$$\operatorname{Re}(p(t)\, \overline{\sigma_1(t)}) > 0, \qquad t \in M(f - p^*)$$
$$\operatorname{Re}(p(t)\, \overline{\sigma_2(t)}) > 0, \qquad t \in B(p^*)$$

does not hold true, which means that the system of inequalities

$$\operatorname{Re}(\mathbf{z}, \mathbf{b}(t)) > 0, \qquad t \in M(f - p^*)$$
$$\operatorname{Re}(\mathbf{z}, \mathbf{c}(t)) > 0, \qquad t \in B(p^*)$$

is incompatible. Due to compactness of the set $\mathscr{B}$ in view of Theorem 2.2 this can happen if and only if the origin of the space $\mathbb{C}^n$ belongs to the convex hull of $\mathscr{B}$. ▌

THEOREM 4.3. *A polynomial $p^* \in P_\Omega$ is a best approximation to $f \in C(Q) \backslash P_\Omega$ from P_Ω if and only if there exist such sets $A_0 = \{t_1, t_2, ..., t_k\} \subset M(f - p^*)$, $B_0 = \{t'_1, t'_2, ..., t'_m\} \subset B(p^*)$ ($k \geqslant 1$, $k + m \leqslant 2n + 1$) and positive constants $\lambda_1, ..., \lambda_k$, $\lambda'_1, ..., \lambda'_m$, that for each polynomial $p \in P$ the following condition holds*:

$$\sum_{l=1}^{k} \lambda_\ell\, p(t_\ell)\, \overline{\sigma_1(t_\ell)} + \sum_{s=1}^{m} \lambda'_s\, p(t'_s)\, \overline{\sigma_2(t'_s)} = 0. \tag{4.17}$$

Proof. $\Rightarrow$ Let p^* be a best approximation to f from P_Ω. According to Theorem 4.2, the origin of the space $\mathbb{C}^n$ belongs to a convex hull of $\mathscr{B}$. In view of Carathéodory's theorem one can find such k vectors $\mathbf{b}(t_\ell) \in \mathscr{B}$, $t_\ell \in M(f-p^*)$, $(\ell = 1, ..., k)$, m vectors $\mathbf{c}(t'_s) \in \mathscr{B}$, $t'_s \in B(p^*)$, $(s = 1, ..., m)$ and positive numbers λ_ℓ $(\ell = 1, ..., k)$, λ'_s $(s = 1, ..., m)$ that

$$\sum_{l=1}^{k} \lambda_\ell + \sum_{s=1}^{m} \lambda'_s = 1,$$

$$\sum_{l=1}^{k} \lambda_\ell \mathbf{b}(t_\ell) + \sum_{s=1}^{m} \lambda'_s \mathbf{c}(t'_s) = 0, \tag{4.18}$$

$$k + m \leqslant 2n + 1.$$

We multiply the second of the equalities (4.18) by an arbitrary vector $t = (c_1, ..., c_n) \in \mathbb{C}^n$ and set $p = \sum_{v=1}^{n} c_v \varphi_v$, to obtain (4.17). Let us show that $k \geqslant 1$. Indeed, notice, that for the polynomial p_0 from Hypothesis 2.1 the following condition holds:

$$\operatorname{Re}(p_0(t'_s) - p^*(t'_s))\, \overline{\sigma_2(t'_s)}) > 0, \qquad s = 1, ..., m.$$

Then

$$\sum_{s=1}^{m} \lambda'_s \operatorname{Re}((p_0(t'_s) - p^*(t'_s))\, \overline{\sigma_2(t'_s)}) > 0,$$

or

$$\sum_{s=1}^{m} \lambda'_s (p_0(t'_s) - p^*(t'_s))\, \overline{\sigma_2(t'_s)} \neq 0.$$

$\Leftarrow$ Assume that for some collections $\{t_1, ..., t_k\} \subset M(f-p^*)$, $\{t'_1, ..., t'_m\} \subset B(p^*)$, and positive constants λ_ℓ $(\ell = 1, ..., k)$, λ'_s $(s = 1, ..., m)$ and arbitrary $p \in P$ (4.17) holds. This immediately entails the equality

$$\sum_{\ell=1}^{s} \operatorname{Re}(p(t_\ell)\, \overline{\sigma_1(t_\ell)}) + \sum_{s+1}^{m} \lambda'_s \operatorname{Re}(p(t'_s)\, \overline{\sigma_2(t'_s)}) = 0.$$

Thus, at least either of the numbers

$$\operatorname{Re}(p(t_\ell)\, \overline{\sigma_1(t_\ell)}) \quad (\ell = 1, ..., k) \qquad \text{and} \qquad \operatorname{Re}(p(t'_s)\, \overline{\sigma_2(t'_s)}) \quad (s = 1, ..., m)$$

is non-positive. But then, obviously, the condition (4.14) holds and p^* by Theorem 4.1 is a best approximation to f from P_Ω. This completes the proof. ▮

Remark 4.1. Under the conditions of Theorem 4.3,

$$|A_0 \cup B_0| \leqslant 2n+1-|A_0 \cap B_0|.$$

Remark 4.2. If P is a Haar space and $f \in C_a(Q)\backslash P_\Omega$, the sets A_0 and B_0 in Theorem 4.3 in addition satisfy the condition $|A_0 \cup B_0| \geqslant n+1$.

Indeed, it is easy to show that for the sets A_0, B_0 in Theorem 4.3 the ordered pair $(A_0; B_0)$ is an a.p. of finite sets. Which, in view of Remark 3.1, contains at least one m.a.p. $(A_0'; B_0')$ for f. Taking into account Theorem 3.3, we get

$$|A_0 \cup B_0| \geqslant |A_0' \cup B_0'| \geqslant n+1.$$

Remark 4.3. All the results of this paper remain valid for some weakened system of restrictions Ω, which can be defined as follows. Let X be some open subset of Q; then

$$\Omega_t := \begin{cases} \{z \in \mathbb{C} \mid |z-u(t)| \leqslant r(t), \; t \in Q\backslash X\} \\ \mathbb{C}, \qquad t \in X. \end{cases}$$

Moreover, the functions u and r are continuous on $Q\backslash X$. In addition, the function r is positive on $Q\backslash X$.

Then, by letting $X = Q$ (i.e., there are no restrictions), we obtain as a consequences classical theorems of characterization of best approximation for unrestricted approximation. Let us formulate them.

THEOREM 4.4 [8]. *A polynomial $p^* \in P$ is a best approximation to a function $f \in C(Q)$ if and only if for each $p \in P$ the following conditional inequality holds true*;

$$\min_{t \in M(f-p^*)} \operatorname{Re}(p(t)\,\overline{\sigma_1(t)}) \leqslant 0.$$

THEOREM 4.5 [9–11]. *A polynomial $p^* \in R$ is a best approximation to $f \in C(Q)\backslash P$ form P if and only if there exist such sets $A_0 = \{t_1, ..., t_k\} \subset$*

$M(f-p^*)$ $(1 \leqslant k \leqslant 2n+1)$ and positive constants $\lambda_1, ..., \lambda_k$ that for each polynomial $p \in P$ the following condition holds:

$$\sum_{\ell=1}^{k} \lambda_\ell \, p(t_\ell) \, \overline{\sigma_1(t_\ell)} = 0.$$

5. UNIQUENESS AND STRONG UNIQUENESS OF BEST APPROXIMATION

We assume throughout this section that P is a Haar space.

THEOREM 5.1 (Uniqueness Theorem). *Each function $f \in C_a(Q)$ has a unique best approximation in P_Ω.*

Proof. If $f \in P_\Omega$, the statement of the theorem is obvious. Let $f \in C_a(Q) \backslash P_\Omega$. Assume, that f has in P_Ω two best approximations p_1 and p_2. Then, as it is known, the polynomial $p^* = 1/2(p_1 + p_2) \in P_\Omega$ is also a best approximation for f. Using standard techniques, we get the inclusions

$$\begin{aligned} M(f-p^*) \subset M(f-p_1) \cap M(f-p_2) \subset Z(p_1 - p_2), \\ B(p^*) \subset B(p_1) \cap B(p_2) \subset Z(p_1 - p_2). \end{aligned} \tag{5.19}$$

Consider now an arbitrary m.a.p. $(A_0; B_0)$ for the function f. By virtue of Theorems 3.1 and 3.3 we have

$$A_0 \subset M(f-p^*), \qquad B_0 \subset B(p^*) \tag{5.20}$$

and also

$$|A_0 \cup B_0| \geqslant n+1. \tag{5.21}$$

The inclusions (5.19) and (5.20) along with the inequality (5.21) entail the estimation

$$|Z(p_1 - p_2)| \geqslant |M(f-p^*) \cup B(p^*)| \geqslant |A_0 \cup B_0| \geqslant n+1,$$

which, in view of Definition 2.1, gives $p_1 = p_2$. This completes the proof. ▮

Let us show that for the functions $f \in C(Q) \backslash C_a(Q)$ Theorem 5.1, in general, is incorrect.

EXAMPLE. Let $Q=[0,1]$, $u(t)=0$, $r(t)=1/2$, $\phi_1(t)=1$, $\phi_2(t)=t$, $f(t)=1/2+3/2t$, $t\in[0,1]$. Note, that for each $p\in P_\Omega$ for $t=1$,

$$|\operatorname{Re} p(1)|\leqslant|p(1)|=|p(1)-u(1)|\leqslant r(1)=1/2.$$

Using this, we have

$$E(f)=\inf_{p\in P_\Omega}\max_{t\in[0,1]}|f(t)-p(t)|\geqslant\inf_{p\in P_\Omega}|f(1)-\operatorname{Re} p(1)|\geqslant 3/2.$$

While for the functions $p_1=\phi_1\in P_\Omega$, $p_2=1/2\phi_2\in P_\Omega$ we have

$$\|f-p_1\|=\|f-p_2\|=3/2.$$

Hence, $E(f)=3/2$ and f has in P_Ω two best approximations p_1 and p_2 (besides, $p_1\neq p_2$).

THEOREM 5.2. (Strong Uniqueness Theorem). *Let $p^*\in P_\Omega$ be a best approximation to a function $f\in C_a(Q)$ from P_Ω. Then there exists such a constant $\gamma=\gamma(f)>0$ that any polynomial $p\in P_\Omega$ satisfies the inequality*

$$\|f-p\|^2\geqslant\|f-p^*\|^2+\gamma\,\|p^*-p\|^2. \tag{5.22}$$

Proof. If $f\in P_\Omega$, then the inequality (5.22) is trivial for $\gamma\leqslant 1$. Let $f\in C_a(Q)\backslash P_\Omega$. Then due to Theorem 4.3 and Remark 4.2 there exist such sets $A_0=\{t_1,...,t_k\}\subset M(f-p^*)$, $B_0=\{t'_1,...,t'_m\}\subset B(p^*)(|A_0\cup B_0|\geqslant n+1)$ and positive constants λ_ℓ $(\ell=1,...,k)$, λ'_s $(s=1,...,m)$ that for each polynomial $p\in P$ (4.17) holds. Without loss of generality, we shall assume that

$$\sum_{l=1}^{k}\lambda_l=1. \tag{5.23}$$

For each $p\in P$ set

$$\Lambda_2(p):=\left(\sum_{\ell=1}^{k}\lambda_\ell\,|(p(t_\ell)|^2+\sum_{s=1}^{m}\lambda_s\,|p(t'_s)|^2\right)^{1/2}.$$

It is easy to check that $\Lambda_2(\cdot)$ is a norm on P. Hence, there exists such a constant $\gamma>0$ that for all $p\in P$ the following inequality holds:

$$\Lambda_2^2(p)\geqslant\gamma(\|p\|^2). \tag{5.24}$$

Taking into account (4.17), (5.23) and (5.24), we get

$$\|f-p\|^2 \geqslant \sum_{l=1}^{k} \lambda_\ell \, |f(t_\ell)-p(t_\ell)|^2 + \sum_{s=1}^{m} \lambda'_s \, |u(t'_s)-p(t'_s)|^2 - \sum_{s=1}^{m} \lambda'_s r^2(t'_s)$$

$$= \sum_{l=1}^{k} \lambda_\ell \, |f(t_\ell)-p^*(t_\ell)|^2 + 2 \sum_{l+1}^{k} \lambda_\ell \operatorname{Re}((p^*(t_\ell)-p(t_\ell)\, \overline{\sigma_1(t_\ell)}))$$

$$+ \sum_{l=1}^{k} \lambda_\ell \, |p^*(t_\ell)-p(t_\ell)|^2 + \sum_{s=1}^{m} \lambda'_s \, |u(t'_s)-p^*(t'_s)|^2$$

$$+2 \sum_{s=1}^{m} \lambda'_s \operatorname{Re}((p^*(t'_s)-p(t'_s))\, \overline{\sigma_2(t'_s)}) + \sum_{s=1}^{m} \lambda'_s \, |p^*(t'_s)-p(t'_s)|^2$$

$$- \sum_{s=1}^{m} \lambda'_s \, |u(t'_s)-p^*(t'_s)|^2 = \|f-p\|^2 + \Lambda_2^2(p^*-p)$$

$$\geqslant \|f-p^*\|^2 + \gamma \, \|p^*-p\|^2. \quad \blacksquare$$

Define on the set $C_a(Q)$ the *operator of best approximation* τ, which assigns to each function $f \in C_a(Q)$ its unique best approximation in P_Ω.

THEOREM 5.3. *The operator τ is continuous in $C_a(Q)$.*

Proof. Fix an arbitrary function $f_0 \in C_a(Q)$ and the corresponding constant of strong uniqueness $\gamma = \gamma(f_0)$ in (5.22). Let us show now that for some $\gamma_1 > 0$ and all such $f \in C_a(Q)$ that $\|f-f_0\| \leqslant 1$ the inequality

$$\|\tau(f)-\tau(f_0)\| \leqslant \gamma_1 \, \|f-f_0\|^{1/2},$$

holds, which immediately implies the Lipschitz continuity (with the index 1/2) of the operator τ at the point f_0. Taking into account (5.22), we get

$$\|\tau(f)-\tau(f_0)\| \leqslant \gamma^{-1/2}(\|f_0-\tau(f)\|^2 - \|f_0-\tau(f_0)\|^2)^{1/2}$$

$$\leqslant \gamma^{-1/2}(\|f_0-f\| + \|f-\tau(f)\|^2 - \|f_0-\tau(f_0)\|^2)^{1/2}$$

$$\leqslant \gamma^{-1/2}((\|f_0-f\| + \|f-\tau(f)\|)^2 - \|f_0-\tau(f_0)\|^2)^{1/2}$$

$$\leqslant \gamma^{-1/2}((2\,\|f_0-f\| + \|f_0-\tau(f_0)\|)^2 - \|f_0-\tau(f_0)\|^2)^{1/2}$$

$$= \gamma^{-1/2}(4\,\|f_0-f\|\,(\|f_0-f\| + \|f_0-\tau(f_0)\|))^{1/2}$$

$$\leqslant \gamma_1 \, \|f_0-f\|^{1/2},$$

where $\gamma_1 = 2\gamma^{-1/2}(1+E(f_0))^{1/2}$. $\blacksquare$

Remark 5.1. Theorem 5.2 suggests the standard form of the inequality of strong uniqueness (see [13]) in the complex case. Indeed, set $\gamma_1 = 1/4\gamma$, $\delta = 2\gamma^{-1/2}$. Then for all such $p \in P_\Omega$ that $\| p - p^* \| \leqslant \delta$ we have the following inequality

$$\| f - p \| \geqslant \| f - p^* \| + \gamma_1 \, \| p - p^* \|^2.$$

6. CONCLUDING REMARKS

1. Helly's theorem in the problems of best approximation has been applied by Shnirelman [14], Rademacher and Schoenberg [12] and others.

2. All the statements of this paper (except Theorem 3.3, Remark 4.2 and the theorems of Section 5) are also valid for the case of Q being a compact Hausdorff space. But the existence on the compact Q a Haar space brings very serious conditions on Q (for the real-valued case see Mairhuber [15] and the complex-valued one—Schoenberg and Yang [16] and Overdeck [17]).

ACKNOWLEDGMENTS

The authors acknowledge with gratitude stimulating discussions on the subject of this work with Victor Konovalov and Igor Shevchuk and are thankful to Ram Murty for his support, encouragement and reading of the manuscript.

The research was supported in part by the National Science and Engineering Research Council of Canada.

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